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Engineering Notes

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Global Positioning System-Based Attitude Determination and the Orthogonal Procrustes Problem

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Introduction

THE global positioning system (GPS) has demonstrated its utility not only for position measurement but also for attitude measurement because GPS-based attitude sensors are economical and drift free. Using the GPS to measure attitude requires solving two problems: initialization, that is, carrier-phase cycle ambiguity resolution, and attitude determination from differential range measurements. The focus of this paper is on the second problem. Specifically, it is demonstrated how attitude can be resolved when the GPS antennas in the antenna array are coplanar, but the unit vectors to the GPS satellites are not.

To determine the attitude of a vehicle relative to some fixed reference frame (hereafter referred to as the inertial or I frame), direction or vector measurements of objects whose relative locations are known in the inertial frame can be used. Estimating attitude from vector measurements is known as Wahba's problem[1] and has a closed-form solution.

When attitude is calculated using the GPS, the distances between pairs of antennas mounted in a vehicle-fixed array along unit vectors to GPS satellites are measured using the GPS carrier-phase measurements. In other words, measurements are projected distances or differential range measurements in known directions. A pair of antennas in the array forms a baseline, and each baseline vector is known in the vehicle's local frame (hereafter referred to as the vehicle or V frame). Differential range measurements can be viewed as measurements of the cosines of the angles between the baselines (known in vehicle coordinates) and the unit vectors to the GPS satellites (known in inertial coordinates). Mathematically,

$$r_{i,j} = \left(\boldsymbol{b}_i^V\right)^T \left({}^V\!\boldsymbol{A}^I\right)\left(\hat{\boldsymbol{s}}_j^I\right) + \nu_{i,j} \tag{1}$$

where $r_{i,j}$ is the differential carrier-phase range measurement along baseline i to satellite j, $\hat{\boldsymbol{s}}_j^I$ is the line-of-sight unit vector to satellite j coordinatized in the inertial or I frame, $\boldsymbol{b}_i^V$ is baseline vector i coordinated in the V frame, and $\nu_{i,j}$ is measurement noise. The superscripts I or V will denote coordinatization in the I or V frame, respectively. ${}^V\!\boldsymbol{A}^I$ is an orthonormal transformation from the I to the V frame. For ease of notation, ${}^V\!\boldsymbol{A}^I$ will be denoted by $\boldsymbol{A}$. The T superscript denotes the transpose operator, and the caret indicates unit length. There are m baselines and n satellites. Note that a vector or matrix will be written in boldface, whereas scalars will not be.

Previous GPS attitude determination research has focused on a variety of related topics including initialization, major error sources such as multipath, and attitude determination algorithms. Cohen, in Ref. 2, gives a good introduction to the subject. He outlines the GPS attitude problem and illustrates how a least-squares approach could be used to solve iteratively for the vehicle's attitude. Another algorithm developed by Cohen will be discussed later. Other authors[3–10] used the numerous mathematical properties of the attitude determination problem to apply different solution techniques that, in some cases, incorporated platform dynamics.

Orthogonal Procrustes Problem

It will be shown that determination of attitude from GPS carrier-phase measurements can be formulated as an "orthogonal Procrustes problem" (see Ref. 11). Briefly, the orthogonal Procrustes problem seeks a solution to

$$\min_A \|\boldsymbol{X}_1 - \boldsymbol{X}_2\boldsymbol{A}\|_F^2 \qquad \text{subject to} \qquad \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I} \tag{2}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Equation (2) is equivalent to maximizing $\text{tr}(\boldsymbol{A}^T\boldsymbol{X}_2^T\boldsymbol{X}_1)$, which is accomplished by computing the singular-value decomposition (SVD) of $\boldsymbol{X}_2^T\boldsymbol{X}_1$ so that

$$\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \text{SVD}\left(\boldsymbol{X}_2^T\boldsymbol{X}_1\right) \tag{3}$$

The solution matrix $\boldsymbol{A}$ is then formed as $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{V}^T$. If desired, $\boldsymbol{A}$ can be forced to be proper or right-handed when set to $\boldsymbol{A} = \boldsymbol{U}^+\boldsymbol{V}^{+T}$, where, for any 3×3 orthonormal $\boldsymbol{X}$,

$$\boldsymbol{X}^+ \equiv \boldsymbol{X}\begin{bmatrix} 1 & & \\ & 1 & \\ & & \det(\boldsymbol{X}) \end{bmatrix} \tag{4}$$

Attitude Determination and the Orthogonal Procrustes Problem

The nm differential range measurements in Eq. (1) can be collected into an $m \times n$ measurement matrix $\boldsymbol{R}$, where element $r_{i,j}$ is the differential range measurement along baseline i to satellite j. The m baseline vectors can be collected into a $3 \times m$ baseline matrix $\boldsymbol{B}^V$, and the n line-of-sight unit vectors to the spacecraft can be collected into a $3 \times n$ satellite matrix $\boldsymbol{S}^I$. The problem of determining $\boldsymbol{A}$ can then be posed as

$$\min_A \|\boldsymbol{R} - (\boldsymbol{B}^V)^T\boldsymbol{A}\boldsymbol{S}^I\|_F^2 \qquad \text{subject to} \qquad \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I} \tag{5}$$

For ease of notation, the frame designations superscripts I and V will be dropped for the remainder of this analysis.

Determination of attitude from GPS carrier-phase measurements can be transformed into an orthogonal Procrustes problem in one of two ways. In the first method, $\boldsymbol{R} - \boldsymbol{B}^T\boldsymbol{A}\boldsymbol{S}$ is postmultiplied by the pseudoinverse $\boldsymbol{S}^T(\boldsymbol{S}\boldsymbol{S}^T)^{-1}$, denoted by $\boldsymbol{S}^\dagger$ (Ref. 12). If $\boldsymbol{S}$ is full rank, Eq. (5) is altered to become

$$\min_A \|\boldsymbol{R}\boldsymbol{S}^\dagger - \boldsymbol{B}^T\boldsymbol{A}\|_F^2 \qquad \text{subject to} \qquad \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I} \tag{6}$$

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Note that the solutions to Eqs. (5) and (6) are not the same. The closed-form solution to Eq. (6) is given by

$$G \equiv BRS^{\dagger} \tag{7}$$

$$U\Sigma V^T = \mathrm{SVD}(G) \tag{8}$$

$$A = U^{+}V^{+T} \tag{9}$$

In the second method, a solution is sought to the transposed problem

$$\min_A \|R^T - S^T A^T B\|_F^2 \qquad \text{subject to} \qquad A^T A = I \tag{10}$$

Now, $R^T - S^T A^T B$ is postmultiplied by the pseudoinverse of B (provided that B is full rank), and A^T is solved for by

$$U\Sigma V^T = \mathrm{SVD}(SR^T B^{\dagger}) \tag{11}$$

$$A^T = U^{+}V^{+T} \tag{12}$$

The important difference between these two solutions lies in which matrix must be full rank. The first method [Eqs. (7–9)] requires S to be full rank so that S spans three dimensions. In the absence of a full-rank constraint on B, the GPS antennas can be coplanar when solving for attitude using the first method, an innovative feature not previously known to a closed-form GPS attitude solution. The second method [Eqs. (11) and (12)] requires B to be full rank. This second method could be applied when the baselines are noncoplanar and satellite geometry is poor.

Because the original problem specified in Eq. (5) was altered differently in each of the two methods, the cost functions to be minimized differ, as do the resulting attitude solutions. Therefore, if both S and B are full rank, both methods can be applied but will yield different solutions. Note that vehicle attitude should be resolvable if there are only three baseline antennas and two GPS satellites provided one knows which side of the array the satellites are on. Neither of the two solutions presented can solve for attitude in this scenario.

The solution derived by Cohen et al.[13,14] merits special mention. Although the authors derived their solution differently, it will be shown that their solution method simplifies to the second method outlined in Eqs. (11) and (12). Briefly, Cohen et al.[13] rewrote the GPS attitude problem in a manner similar to Wahba's problem[1] as

$$\min_A \left\| W_B^{\frac{1}{2}}[R - B^T A S] W_S^{\frac{1}{2}} \right\|_F^2 \qquad \text{subject to} \qquad A^T A = I \tag{13}$$

W_B and W_S were symmetric baseline and satellite weighting matrices. W_B was set to $V_B \Sigma_B^{-2} V_B^T$, where V_B and Σ_B come from the SVD of $B = U_B \Sigma_B V_B^T$. All singular values of B were necessarily constrained to be greater than zero, and therefore, the baselines had to be noncoplanar. After some manipulation, it was shown that the SVD of $BW_B RW_S S^T = U\Sigma V^T$ provided a closed-form solution in $A = UV^T$. If W_S is set to the identity matrix, then $BW_B RW_S S^T$ simplifies to $B^{\dagger T} RS^T$, which is the transpose of the matrix to be decomposed in Eq. (11).

Sensitivity Analysis

The sensitivity analysis presented here, a precursor to the subsequent covariance analysis, begins by closely following the method first outlined by Markley.[15] Briefly, as the change in attitude from the inertial to the vehicle frame approaches zero, the angular perturbations to the true attitude vector can be expressed as

$$[\Theta\times] = -(\delta A)A^T \tag{14}$$

$$[\Theta\times] = \begin{bmatrix} 0 & -\Theta_3 & \Theta_2 \\ \Theta_3 & 0 & -\Theta_1 \\ -\Theta_2 & \Theta_1 & 0 \end{bmatrix} \tag{15}$$

Markley shows that

$$\Theta = U^{+}D^{-1}z \tag{16}$$

where

$$D = \begin{bmatrix} \Sigma_2 + \gamma\Sigma_3 & & \\ & \Sigma_1 + \gamma\Sigma_3 & \\ & & \Sigma_1 + \Sigma_2 \end{bmatrix} \tag{17}$$

Σ_i is the ith singular value of G defined by Eq. (7) as $BRS^{\dagger}$ or by Eq. (11) as $SR^T B^{\dagger}$ and $\gamma \equiv \det(U)\det(V)$. The vector z is defined by

$$[z\times] \equiv [U^{+T}(\delta G)V^{+}]^T - U^{+T}(\delta G)V^{+} \tag{18}$$

Equation (18) can be rewritten in vector form as

$$z = \sum_{i=1}^{3} -\left[V^{+T}\hat{e}_i\times\right]U^{+T}\delta G\hat{e}_i \tag{19}$$

where $\hat{e}_i$ is a unit vector in the ith direction. Equation (16) becomes

$$\Theta = -U^{+}D^{-1}\sum_{i=1}^{3}\left[V^{+T}\hat{e}_i\times\right]U^{+T}\delta G\hat{e}_i \tag{20}$$

Equation (20) approximates the attitude sensitivity to perturbations in G.

Covariance Analysis

If the expected values of δG are zero, then, to first order, the expected value of Θ is also zero, and the attitude estimate is unbiased. The first-order approximation to the covariance of the attitude estimate error P_Θ is then given by

$$P_\Theta = E\{\Theta\Theta^T\} \tag{21}$$

$$P_\Theta = \sum_{i=1}^{3}\sum_{j=1}^{3} E\left\{\begin{array}{l} U^{+}D^{-1}\left[V^{+T}\hat{e}_i\times\right]U^{+T}\delta G\hat{e}_i \\ \quad\times\,\hat{e}_j^T\delta G^T U^{+}\left[V^{+T}\hat{e}_j\times\right]^T D^{-1}U^{+T} \end{array}\right\} \tag{22}$$

$$P_\Theta = \sum_{i=1}^{3}\sum_{j=1}^{3} E\left\{\begin{array}{l} U^{+}D^{-1}\left[V^{+T}\hat{e}_i\times\right]U^{+T}M \\ \quad\times\,U^{+}\left[V^{+T}\hat{e}_j\times\right]^T D^{-1}U^{+T} \end{array}\right\} \tag{23}$$

where

$$M_{i,j} \equiv \delta G\hat{e}_i\hat{e}_j^T\delta G^T \tag{24}$$

and E denotes the expectation. What value to use for M depends on the statistics of δG.

Note that, if either B or S is not full rank, then information will be lost when computing P_Θ because some input noise directions will generate no output because G will be of rank two. Therefore, Eq. (23) should be applied only when both B and S are full rank. As an example, consider the case when B is of rank two, that is, the baselines are coplanar, and δG is dominated by measurement noise so that δB and δS can be assumed to be negligible. For the first method, $\delta G \cong B\delta RS^{\dagger}$, and therefore,

$$M_{i,j} = B\delta RS^{\dagger}\hat{e}_i\hat{e}_j^T S^{\dagger T}\delta R^T B^T \tag{25}$$

$$M_{i,j} = B\delta RT_{i,j}\delta R^T B^T \tag{26}$$

$$T_{i,j} \equiv S^{\dagger}\hat{e}_i\hat{e}_j^T S^{\dagger T} \tag{27}$$

Furthermore, suppose that the measurement noise between any satellite-baseline pair can be assumed to be uncorrelated with any other. For this particular example, the lack of correlation between elements of δR means that $E\{\delta RT_{i,j}\delta R^T\}$ will be diagonal with elements

$$E\left\{\delta RT_{i,j}\delta R^T\right\}_{p,p} = \sum_{k=1}^{n} E\left\{\delta R_{p,k}^2\right\}T_{k,k} \tag{28}$$

Although $\delta \boldsymbol{R} \boldsymbol{T}_{i,j} \delta \boldsymbol{R}^T$ could be full rank, multiplication by a rank-deficient $\boldsymbol{B}$ in Eq. (26) results in a loss of information, and Eq. (23) no longer accurately approximates the attitude error covariance. In practical terms for this example, one or more diagonal terms in $\boldsymbol{P}_\Theta$ will appear as zero when in fact correlations do exist. This argument can be extrapolated to realize that, as the condition number of either $\boldsymbol{B}$ or $\boldsymbol{S}$ degenerates, the fidelity of $\boldsymbol{P}_\Theta$ as a true representation of attitude error uncertainty decreases.

Conclusions

The determination of attitude from differential GPS range measurements was posed as an orthogonal Procrustes problem so that a closed-form solution could be found. It was shown that the attitude determination problem could be solved in not one but two different methods. The first method allowed for coplanar baselines, whereas the second method allowed for planar satellite geometry. Subsequent sensitivity and covariance analysis showed that both solutions yielded unbiased estimates. For small attitude changes, a general expression was derived for the covariance of the attitude solution that showed how poor geometry could adversely affect solution quality.

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New Proportional Navigation Law for Ground-to-Air Systems

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I. Introduction

GUIDANCE applications in ground-to-air systems are characterized by high requirements in terms of small miss distances against fast moving targets. Modern defense systems are typically equipped with highly sophisticated subsystems, both onboard the intercepting missiles and as part of the supporting systems, for example, radar systems, enabling good estimates of the interception time to go. Guidance methods that exploit such estimates are more likely to become candidates for realistic applications, whereas simple proportional-navigation (PN) guidance will usually fall short with respect to satisfying the tough requirements.

It is well known that PN (with $N' = 3$) is, in fact, an optimal strategy for the linearized problem,[1,2] when the cost J is the control effort, as follows:

$$J = \int_0^{t_f} u^2(t)\, \mathrm{d}t \tag{1}$$

where u is the missile's lateral acceleration and t_f is the collision time (the elapsed time from the beginning of the end game till interception).

Improved guidance schemes can be obtained with an appropriate selection of a cost function that replaces J (see Ref. 2). This paper employs an exponential term in J. (See Refs. 3 and 4 for exponentially weighted quadratic performance index.)

We let

$$J = \int_0^{t_f} e^{-k(t_f - t)} u^2(t)\, \mathrm{d}t \tag{2}$$

The motivation for this cost is for guiding aerodynamically maneuvering ground-to-air missiles that loose their maneuverability as the air density decreases. Because the air density can be approximated by an exponential term (as a function of altitude), it makes sense to weigh maneuvers in this manner, whereby the penalty for late maneuvers is higher than for earlier ones. As will be shown, this cost leads to a new PN law with a time-varying navigation gain.

The paper is organized as follows: In the next section the standard two-dimensional problem geometry and the mathematical modeling will be reviewed. The problem formulation and analysis by the optimal control theory will be presented in Sec. III. In Sec. IV some numerical results are presented, and in Sec. V the Note is summarized.

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